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TRANSONIC FLOW OF AN ELASTIC MEDIUM PAST A THIN SOLID*

I.V. SIMONOV

A plane problem of the steady state of a body in an infinite elastic medium in the range of sonic velocities is considered. The generalized Hilbert problem arises for the complex function determining the longitudinal part of the velocity and stress field, and the transverse part of the field is expressed simply by the solution of the Hilbert problem. The separation of the medium from the body contour at the trailing edge is computed. In the former case the position of the separation point is not known, and the method of fixing this point differs from that in /1/ where the problem of wedging is considered at sub-Rayleigh velocities. In /1/ the free surface is formed before the frontal part of the wedge and the separation point is found from the condition that the stresses are finite. In the present problem, just as in the case of super-Rayleigh subsonic motion of a wedge /2, 3/, the condition that the stresses are finite (and even continuous) at the separation point is ensured by the solution beforehand, and a more accurate analysis is required, which will include, to clarify the problem, the computation of the first few terms of the asymptotic expansion of the solution near the separation point. The separation point is fixed using the condition of attachment of the flow in the zone of contact, and the condition of impermeability of the region between the separation point and the trailing edge of the body. The demand that both these physical conditions are met locally near the point of

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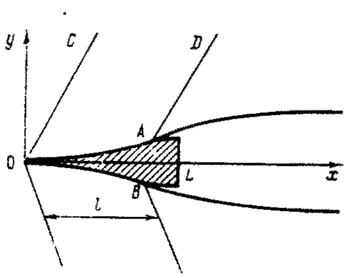
detachment, leads to an equation for determining the size of the region of contact. Satisfying this equation is equivalent to the condition that the curvature of the cavity surface is bounded at the point of contact, the condition known in hydrodynamics as the Brillouin-Villat condition /4/. Certain generalizations of this condition and of the problem as a whole (taking into account the friction, three-dimensionality and non-stationarity) are discussed. The supersonic wedging was studied in /5, 6/.

1. Formulation of the problem. A thin, plane, perfectly rigid symmetric body of width L (a blade) is placed without friction in an infinite flow of an elastic medium. At infinity, ahead of the frontal part of the blade ($x\to-\infty$, $|y|<\infty$) (see the figure) the elastic medium is load-free, and the stream velocity parallel to the axis of the body is equal to c and takes values in the interval $c_2<c<c_1$ (c_1 and c_2 denote the longitudinal and transverse wave velocities of the medium). The problem consists of constructing a steady velocity and stress field in an elastic medium within the framework of the following scheme of flow. The blade is in complete contact with the medium near the sharp leading edge on the segment AOB ($0<x<l$), the surface of the elastic medium behind the points A and B is stress-free, and a cavity is formed (see fig.). In a special case $l=L$. The scheme, analogous in many aspects to the scheme of flows in the subsonic, super-Rayleigh range of velocities /2, 3/, is substantiated below.



The symmetry principle enables us to formulate the problem in the upper half-plane $y>0$ only. The boundary conditions transferred to the boundary $y=0$, take the form

$$v=0\ (-\infty<x\leqslant 0),\ v=f(x)\ (0\leqslant x\leqslant l) \tag{1.1}$$
$$\sigma_y=0\ (l<x<\infty),\ \tau_{xy}=0\ (-\infty<x<\infty)$$

The following notation is adopted: σ_x, σ_y, τ_{xy} are the stress tensor components, and (u, v) and (U, V) are the perturbed displacement and velocity vectors respectively of the points of the elastic medium. The following constraints are imposed on the function $f(x)$ representing the profile of the streamlined body (a prime denotes ordinary differentiation):

$$f(0)=0,\ |f'(x)|\ll 1,\ |f'''(x)|<\infty\ (0\leqslant x\leqslant L) \tag{1.2}$$

In addition to (1.1), the following extra conditions will be taken into account in the course of the solution:

1) the flow is attached to the segment OA: the stress σ_y must not be tensile

$$\sigma_y(x, 0)\leqslant 0\ (0<x<l) \tag{1.3}$$

2) the cavity edge does not intersect the body contour behind the separation point A

$$v(x)\geqslant f(x)\ (l\leqslant x\leqslant L) \tag{1.4}$$

3) the energy density is integrable within any finite volume of the elastic medium;
4) the condition of radiation (this will become clear in the course of the solution).

In the plane steady-state problem of the theory of elasticity all functions can be expressed in terms of the displacement potentials φ and ψ by means of the formulas

$$\begin{gathered} u=\varphi_{,x}+\psi_{,y},\ v=\varphi_{,y}-\psi_{,x} \\ U=u_{,x}=\varphi_{,xx}+\psi_{,xy},\ V=v_{,x}=\varphi_{,xy}-\psi_{,xx} \\ \sigma_x=\alpha\varphi_{,xx}+\psi_{,xy},\ \sigma_y=-\beta\varphi_{,xx}-\psi_{,xy},\ \tau_{xy}=\varphi_{,xy}-\beta\psi_{,xx} \\ \beta=1-c^2/(2c_2^2)<{}^1/_2,\ \alpha=2-\beta-c^2/c_1^2 \end{gathered} \tag{1.5}$$

where the stresses are normalized to 2μ (μ is the shear modulus), and the velocities to c.

The problem of determining φ and ψ is formulated as follows. In the region ($-\infty<x<\infty$, $0<y<\infty$) the functions must satisfy the elliptic and wave equations

$$\beta_1^2\varphi_{,xx}+\varphi_{,yy}=0,\quad \beta_2^2\psi_{,xx}-\psi_{,yy}=0 \tag{1.6}$$
$$\left(\beta_j^2=\left|1-\frac{c^2}{c_j^2}\right|,\quad j=1, 2\right)$$

At $y=0$ they must satisfy the boundary conditions following from (1.1), (1.2)

$$\begin{gathered} \beta\varphi_{,xx}+\psi_{,xy}=0\quad (x>l),\quad \varphi_{,xy}-\beta\psi_{,xx}=0\quad (|x|<\infty) \\ \varphi_{,xy}-\psi_{,xx}=f'(x)\quad (0\leqslant x\leqslant l),\quad \varphi_{,xy}-\psi_{,xx}=0\quad (x<0) \end{gathered} \tag{1.7}$$

The general solution of the first equation of (1.6) is a harmonic function of the stretched coordinates x, $\beta_1 y$, while that of the second equation is D'Alembert's solution. For this reason it is best to replace φ and ψ by new unknown functions Φ and Ψ

$$\Phi(z) = \varphi_{,xx} - i\beta_1^{-1}\varphi_{,xy}, \quad \Psi(z_1) = \Psi_1''(z_1), \quad \Psi_1(z_1) = \psi(x, y) \tag{1.8}$$

Here $\Phi(z)$ is an analytic function of the complex variable $z = x + i\beta_1 y$ (the second derivative in z of the complex displacement potential), $\Psi_1(z_1)$ is a piecewise-smooth D'Alembert's solution chosen taking the radiation condition into account, and there are no transverse waves arriving from infinity, so that $z_1 = x - \beta_2 y$. Conversely, the second derivatives of φ and ψ are connected with the new functions by the following relations:

$$\varphi_{,xx} = \operatorname{Re}\Phi, \quad \varphi_{,xy} = -\beta_1 \operatorname{Im}\Phi, \quad \psi_{,xx} = \Psi, \quad \psi_{,xy} = -\beta_2\Psi \tag{1.9}$$

Here (1.6) will be satisfied and by substituting (1.9) into (1.5) we can express uniquely the stresses and velocities in terms of Φ and Ψ, bypassing the intermediate differentiation or integration.

Let us now formulate the boundary conditions for Φ and Ψ at $y = 0$. We obtain them by substituting (1.9) into (1.7) and eliminating Ψ

$$\begin{gathered}\operatorname{Re}[(\beta^2 - i\beta_1\beta_2)\Phi] = 0, \quad \Psi = -\beta^{-1}\beta_1 \operatorname{Im}\Phi \quad (l < x < \infty) \\ \operatorname{Im}\Phi = Af'(x), \quad \Psi = -(1 + \beta_1 A) f'(x) \quad (0 \leqslant x \leqslant l) \\ \operatorname{Im}\Phi = 0, \quad \Psi = 0 \quad (x < 0), \quad A = \beta[\beta_1(1-\beta)]^{-1}\end{gathered} \tag{1.10}$$

The boundary conditions have become separated. For the analytic function $\Phi(z)$ we now have the generalized Hilbert problem /7, 8/. After solving this problem we can easily recover the function Ψ in terms of $\operatorname{Im}\Phi$ on the real axis, from the conditions (1.10), by replacing x by z_1.

2. Solution of the Hilbert problem. We will seek a solution in the class of functions satisfying the following constraints on their behaviour at the singularities:

$$|\Phi| < \frac{\text{const}}{|z|^{1/2}} \quad (z \to 0, \infty), \quad |\Phi| < \frac{\text{const}}{|z - l|^{1/2}} \quad (z \to l + i\cdot 0) \tag{2.1}$$

which follow from the supplementary conditions (3), (4). The condition at infinity follows from the requirement that the energy flux at infinity must be finite, taking the results of /9/ into account. The solution in the class of functions shown is unique, and can be found using one of the methods given in /7, 8/

$$\Phi(z) = A\frac{(l-z)^\gamma}{\pi}\int_0^l \frac{f'(t)\,dt}{(l-t)^\gamma(t-z)} \quad \left(\gamma = \frac{1}{\pi}\operatorname{arctg}\frac{\beta^2}{\beta_1\beta_2}\right) \tag{2.2}$$

Conditions (2.1) fix, roughly speaking, the choice of the power index in solution (2.2). To separate the single-valued branch of the function $(z - l)^\gamma$, we make a cut along the ray $(l < x < \infty,\ y = 0)$. The function takes positive values on the upper edge of the cut, and $(l - z)^\gamma = e^{-i\pi\gamma}(z - l)^\gamma$.

Let us write the limiting values of the function Φ at the boundary $y = 0$, obtained using the Sokhotskii-Plemel' formulas /7, 8/

$$\begin{gathered}\Phi = A[if'(x) + h(x)] \quad (0 < x < l) \\ \Phi = Ae^{-i\pi\gamma}h(x) \quad (x > l), \quad \Phi = Ah(x) \quad (x < 0) \\ h(x) = \frac{|l - x|^\gamma}{\pi}\int_0^l \frac{f'(t)\,dt}{(l-t)^\gamma(t-x)}\end{gathered} \tag{2.3}$$

where the integral over $0 < x < l$ is regarded as the principal value. Taking into account (2.3) we obtain from (1.10) the final expressions for the function $\Psi(z_1)$

$$\begin{gathered}\Psi = 0 \quad (z_1 < 0), \quad \Psi = (\beta - 1)^{-1}f'(z_1) \quad (0 < z_1 < l) \\ \Psi = (1 - \beta)^{-1}\sin(\pi\gamma)\,h(z_1) \quad (z_1 > l)\end{gathered} \tag{2.4}$$

Using the formulas (1.5), (1.9), (2.2), (2.4) we can easily obtain expressions for the stresses and velocities on the whole half-plane in terms of Cauchy-type integrals. In problems of this type, however, the behaviour of the functions sought at the singularities and at the region boundary is of greatest interest. We shall therefore not compute the final formulas for $\sigma_{xx}, \ldots$, and just make the following comment on the qualitative aspects of the solution. The transverse wave fronts $z_1 = 0, l$ (OC and AD, see the figure) divide the whole half-plane into three subregions. The functions sought are defined in each of these subregions in a different manner, as a result of the fact that the transverse part of the field is described by different formulas according to (2.4). The behaviour of these functions on approaching the subregion boundaries, on the other hand, is determined by the behaviour of the functions $f'(x)$ and $h(x)$.

It is the study of these (and other) features that we shall now attempt.

3. Behaviour of the solution near the singularities $z = 0, l, \infty$ and transverse wave fronts $z_1 = 0, l$. Let us carry out an asymptotic analysis of formulas (2.2), (2.4). It can be shown that the function Φ in a small neighbourhood of the blade tip $z \to 0$, and the function Ψ in the neighbourhood of the front OC $(z_1 \to 0)$ admit of the following asymptotic representations ($H(z_1)$ is the Heaviside function):

$$\Phi = -A\xi\left(\ln\frac{z}{l} - i\pi\right) + C + O\left(z\ln\frac{z}{l}\right) \tag{3.1}$$
$$\Psi = H(z_1)(\beta - 1)^{-1}[\pi\xi + O(z_1)] \quad (\xi = \pi^{-1}f'(0))$$
$$C = \frac{A}{\pi}\int_0^1 \frac{(1-t)^{-\gamma}f'(lt) - \pi\xi}{t}\,dt$$

The stresses and velocities written in a polar system of coordinates $z = lre^{i\theta}$ are equivalent, as $r \to 0$, to the following functions:

$$\sigma_x \sim -\alpha\xi A\ln r + \alpha C, \quad \sigma_y \sim \beta\xi A\ln r - \beta C \tag{3.2}$$
$$\tau_{xy} \sim \beta_1\xi A(\theta - \pi), \quad U \sim -\xi A\ln r + C$$
$$V \sim \beta_1\xi A(\theta - \pi) \quad (\pi \leqslant \theta < \theta_0, \ \theta_0 = \operatorname{arctg}(\beta_1\beta_2^{-1}))$$
$$\tau_{xy} \sim \beta_1\xi A\theta, \quad V \sim \xi(\pi + \beta_1 A\theta) \quad (0 \leqslant \theta < \theta_0)$$

and for the remaining functions, when $0 \leqslant \theta \leqslant \theta_0$, the above expressions are supplemented by constant terms describing the jumps at the front of the transverse wave

$$\Delta\sigma_x = -\Delta\sigma_y = \Delta U = \pi\beta_2\xi(1-\beta)^{-1}$$

From (3.2) it follows that the functions σ_x, σ_y and U have a logarithmic singularity at zero $(\xi \neq 0)$ or they are of the order of unity $(\xi = 0)$, and $\tau_{xy} \sim V = O(1)$ if $\xi \neq 0$ or they are of the order of accuracy of (3.2) equal to $O(r\ln r)$ if $\xi = 0$. The stress σ_y is compressive in a small neighbourhood of zero, and the sign of σ_x depends on the sign of β: the stress is tensile when $\beta > 0$ $(c_2 < c < \sqrt{2}c_2)$ and compressive when $\beta < 0$ $(\sqrt{2}c_2 < c < c_1)$. The degenerate case $\beta = 0$ will be considered separately.

In the neighbourhood of the point at infinity we have

$$\Phi(z) = \frac{C_1}{z^{1-\gamma}} + O\left(\frac{1}{z^{2-\gamma}}\right), \quad \Psi = \frac{C_2}{z_1^{1-\gamma}} + O\left(\frac{1}{z_1^{2-\gamma}}\right) \quad (|z|, z_1 \to \infty)$$
$$C_1 = -\frac{A}{\pi}e^{-i\pi\gamma}\int_0^l \frac{f'(t)\,dt}{(l-t)^\gamma}, \quad C_2 = -\frac{\beta_1}{\beta}\operatorname{Im} C_1$$

The power flux from the longitudinal component of the field at infinity is equal to zero, since $\Phi(z)$ decreases as $z \to \infty$ more rapidly than $z^{-1/2}$ /9/. Since the power flux into other singularities $z = 0, l$ is also zero, the energy balance is as follows: the power necessary to move the blade and equal to cF, where F is a resistance equal to the power radiated at infinity by transverse waves in the corresponding volume at the elastic medium. The resistance is calculated from the formula

$$F = -4\mu\int_0^l \sigma_y(x, 0)f'(x)\,dx$$

and the expression for $\sigma_y(x, 0)$ can be reduced to the form

$$\sigma_y(x, 0) = B(l-x)^\gamma G(x) \quad (0 < x \leqslant l) \tag{3.3}$$
$$B = \frac{\beta^2}{\pi\beta_1(1-\beta)}, \quad G(x) = \int_0^l \frac{f'(t) - f'(x)}{(l-t)^\gamma(x-t)}\,dt - f'(x)\int_{-\infty}^0 \frac{dt}{(l-t)^\gamma(x-t)}$$

In deriving (3.3) we used the formulas (1.5), (1.9), (2.3), (2.4), the method of separating the singularities given in /7, 8/, and the following formulas from /10/ for evaluating the Cauchy-type integrals with a power-type singularity of the density:

$$\int_0^l \frac{dt}{(l-t)^\gamma(z-t)} = \frac{e^{-i\pi\gamma}}{\sin(\pi\gamma)}\frac{\pi}{(l-z)^\gamma} - \int_{-\infty}^0 \frac{dt}{(l-t)^\gamma(z-t)} = \tag{3.4}$$
$$\frac{e^{-i\pi\gamma}}{\sin(\pi\gamma)}\frac{\pi}{(l-z)^\gamma} - \frac{1}{\gamma l^\gamma} + O\left(1 - \frac{z}{l}\right) \quad (z \to l + i\cdot 0)$$

We note that at sub-Rayleigh velocities there is no radiation at infinity and the elastic energy is dissipated near the crack tip moving in front of the wedge /1/. In the super-Rayleigh subsonic mode the energy is radiated, but both transverse and longitudinal component of the field /2/ take part in the process. The form and normal component of velocity of the cavity boundary are found from the formulas obtained using (1.5), (2.3), (2.4)

$$y(x)=v(x,0)=\int_l^x V(x_1,0)\,dx_1+f(l) \qquad (3.5)$$
$$V(x,0)=-\sin(\pi\gamma)\,h(x)\ (x>l)$$

When $x\to\infty$ the asymptotic behaviour of $y(x)$ is as follows:

$$y(x)\sim C_3x^{\gamma},\quad C_3=\frac{\sin(\pi\gamma)}{\pi\gamma}\int_0^l\frac{f'(t)\,dt}{(l-t)^{\gamma}}$$

i.e. the cavity expands to infinity if $\gamma\neq 0\ (\beta\neq 0)$ just as in the subsonic, super-Rayleigh range of velocities /2/ (where $y(x)\sim\mathrm{const}\cdot x^{1/2}$ as $x\to\infty$). Finally, we give the asymptotic formulas describing the behaviour of the function Φ in a small neighbourhood of the separation point $(z\to l+i0)$ and of the function Ψ near the transverse wave front AD (see the figure) emerging from the separation point

$$\Phi=Af'(l)\,[i-\operatorname{ctg}(\pi\gamma)]-\pi^{-1}AG(l)\,(l-z)^{\gamma}+O(l-z) \qquad (3.6)$$
$$\Psi=(\beta-1)^{-1}f'(l)+O(l-z_1)\ \ (z_1\to l-0)$$
$$\Psi=\frac{f'(l)}{\beta-1}+\frac{\sin(\pi\gamma)\,G(l)\,(z_1-l)^{\gamma}}{\pi(\beta-1)}+O(z_1-l)\quad(z_1\to l+0)$$

Formulas (3.6) obtained from (2.2), (2.4) with help of (3.4), show, in particular, that the discontinuity at the front AD is weak: the stresses and velocities and continuous on passing across the line $z_1=l$.

The asymptotic representations of the stresses and velocities when $z\to l+i\cdot 0$, depend on the character of the separation. By analogy with a liquid /4/ we shall distinguish between the cases of sharp and smooth separation. We define, in the assumptions (1.2), the sharp separation as a separation taking place at the trailing edge of the blade, when the following conditions hold:

$$G(x)\leqslant 0\ \ (0<x<L),\ G(L)<0 \qquad (3.7)$$

the first of which represents the condition that the flow is continuous (1.3), written taking (3.3) into account. The asymptotics of the functions, as $z\to L+i0$, are obtained from (1.5), (1.9) and (3.6), taking the second condition of (3.7) into account

$$\sigma_x=\frac{\beta_2(\beta-\alpha)}{\beta(1-\beta)}f'(L)+O(|L-z|^{\gamma}),\ U=-\frac{\beta_2}{\beta}f'(L)+O(|L-z|^{\gamma}) \qquad (3.8)$$
$$V=f'(L)+O(|L-z|^{\gamma}),\ \ \sigma_y\sim\tau_{xy}=O(|L-z|^{\gamma})$$

The sharp separation in the transonic mode of flow (as well as in the subsonic, super-Rayleigh velocities /2, 3/) is characterized by the fact that the functions themselves are continuous at the separation point, and the derivatives experience a discontinuity of the second kind. The cavity profile curvature and the acceleration are not bounded as $x\to l+0,\ y=0$. Analysis of (3.5) shows that the free stream line at the edge bends into the cavity. We shall show for comparison that in the case of sub-Rayleigh velocities the stresses and velocities themselves will be unbounded /1/.

4. Determination of unknown parameter l in the case of smooth separation. The first condition of (3.7) may be violated. In this case the separation occurs when $0<l<L$.

Assertion 1. With the assumptions made (Sect.1) the condition

$$G(l)\equiv\int_0^l\frac{f'(t)-f'(l)}{(l-t)^{1+\gamma}}-\frac{f'(l)}{\gamma l^{\gamma}}=0 \qquad (4.1)$$

is the necessary condition of separation at the intermediate point $0<l<L$.

Proof. Let us satisfy locally the requirement (1.3) to the left of the point A, and condition (1.4) to the right of this point, using the asymptotic expansions of the functions $\sigma(x,0)$ and $V(x,0)$ near the separation point. To satisfy (1.4) locally we must have

$$\int_l^x[V(x_1,0)-f'(x_1)]\,dx_1\geqslant 0\quad(x\to l+0) \qquad (4.2)$$

Using (3.5) and (3.4) we can reduce the integrand in (4.2) to the form

$$V(x, 0) - f'(x) = \pi^{-1} \sin(\pi\gamma)\,(x - l)^{\gamma} G(x) \quad (l \leqslant x \leqslant L) \tag{4.3}$$

When $x \to l \pm 0$, the function $G(x)$ can be written in the form (to do this it is sufficient to satisfy the third condition of (1.2))

$$G(x) = G(l) + O(1 - x/l) \tag{4.4}$$

From (4.3) and (4.4) we find that to ensure inequality (4.2) we must have

$$G(l) \geqslant 0 \tag{4.5}$$

Similarly we can conclude that to satisfy (1.3) in the left neighbourhood $x = l$, taking (3.3) and (3.4) into account, we must have

$$G(l) \leqslant 0 \tag{4.6}$$

Relations (4.5) and (4.6) can hold simultaneously only in the case of equality. This proves the assertion.

When condition (4.1), which we shall call the condition of smooth separation, holds, then O, which is a term in the asymptotic form (3.8), is replaced by $O(|l - z|)$. This follows from (1.5), (1.9), (3.6). From (4.1), (4.3), (4.4) it follows that (the term of order $O(|z - l|)$ in the expansions of the functions $\sigma_y(x, 0)$ and $V(x, 0) - f'(x)$ vanishes)

$$V(x, 0) - f'(x) = O[(x - l)^{1+\gamma}] \quad (x \to l + 0)$$

and we can formulate an assertion equivalent to Assertion 1, namely

Assertion 2. The cavity profile curvature is finite at the point of smooth separation, and equal to the curvature of the streamlined contour. Assertion 2 holds in problems of flows of ideal fluids past solids, is called the Brillouin-Villat condition and is used to obtain the smooth separation points. Thus we can formulate the assertion proved above in a different way: the Brillouin-Villat condition remains valid for the transonic flows of elastic media past thin solids. By the way, the range of flow velocities considered here enables us, by means of the limiting process $c_2 \to 0$ $(\mu \to 0)$, to pass to the case of the flow of an ideal compressible fluid past thin contours. In this case the form of solution for the longitudinal part of the field (function Φ) will remain unchanged and the transverse component of the field will vanish. The coefficients will take the values $2\mu B \to (2\pi\beta_1)^{-1} \rho c^2$ $(\mu \to 0)$, $A = -\beta_1^{-1}$, $\gamma = 1/2$ (ρ is the density), and one of the well-known schemes of fluid flow with infinite cavity past a solid is obtained /4, 11/.

When condition (4.1) is used in practice to find the parameter l, the smallest root of (4.1) must obviously be taken. Condition (1.3) will then hold globally, and to substantiate the scheme used it merely remains to check that the condition of impermeability (1.4) holds (obviously, there may be other points of attachment and separation).

We will consider the degenerate case $\beta = 0$: $c = \sqrt{2} c_2$, $\gamma = 0$, $\beta_2 = 1$ separately. The case differs from that of $\beta \neq 0$ in the fact that the longitudinal part of the field vanishes $(\Phi(z) \equiv 0)$ and the function Ψ takes the form $(z_1 = x - y)$

$$\Psi(z_1) = -H(z_1) H(l - z_1) f'(z_1) \quad (|z_1| < \infty)$$

Correspondingly, the stress and velocity fields are piecewise continuous and take non-zero values in the corridor $0 \leqslant x - y < l$. To find the constant l, we write the expression for σ_y

$$\sigma_y(x, y) = -f'(x - y)$$

which is also valid on the area of contact, and for V and v in the region $z_1 > l$

$$V \equiv 0, \quad v = v(l) = \text{const}$$

are also valid on the cavity surface. The above expressions and conditions (1.3), (1.4) together imply that the parameter l can be chosen uniquely when $\beta = 0$. If $f'(x) > 0$ $(0 < x \leqslant L)$, then $l = L$ and the stresses and velocities undergo a jump at the front $z_1 = L$. Otherwise, l is the smallest root of the equation $f'(x) = 0$; the stresses and velocities are continuous across the front $z_1 = l$ but their derivatives are not, unlike the case $\beta \neq 0$. In both versions the cavity is a half-strip and the solution holds in the second case provided that (the condition of impermeability)

$$f(x) \leqslant f(l) \quad (l \leqslant x \leqslant L)$$

5. Example. Let us consider a class of contours for which

$$f'(x) = ax^2 - bx + d \quad (d \geqslant 0) \tag{5.1}$$

The integrals in (4.1) can be evaluated to completion to yield a quadratic equation in l, the roots of which are

$$l_{1,2} = \frac{2-\gamma}{4a}\left[b \pm \left(b^2 - \frac{8(1-\gamma)}{2-\gamma} ad\right)^{1/2}\right] \tag{5.2}$$

The roots of (5.2) include real and positive roots, provided that the following constraints are imposed on the coefficients:

$$(2-\gamma)\,b^2 \geqslant 8\,(1-\gamma)\,ad,\ b>0 \cap (b<0 \cup a<0)$$

When $a>0, b>0, d\neq 0$ we have two positive roots (a convex-concave profile); when $a<0$ we have a single positive root (the profile is strictly convex if we also have $b>0$, and concave-convex if $b<0$). In every case the smallest positive root is chosen as the position of the separation point. It can be shown that in this case the necessary condition of separation in the form

$$G'(l)>0 \tag{5.3}$$

will hold, and will ensure local satisfaction of the condition that the flow is attached to the left of the separation point, and of the condition of impermeability to the right of the separation point. Actually, the second positive root does not satisfy condition (5.3). The final choice of the scheme of flow past the profile of the form (5.1) is arrived at by comparing the roots of (5.2) with L. If there are no positive roots amongst (5.2), then the flow separates from the edge at $x=L$. If on the other hand l_1 is the smallest positive root, then at $L\leqslant l_1$ the flow pattern is as before, and smooth separation occurs when $L=l_1$. When $l_1<L$, the separation is smooth and takes place at the point $x=l_1$. However, it must be checked that condition (1.4) holds: i.e. that the elastic medium will not collide with the blade behind the separation point when $l_1<x<L$.

6. On certain generalizations of the condition of smooth separation, and of the problem as a whole. Assertion 1 can be generalized as follows to the case of three-dimensional non-steady separation of the elastic medium from a solid based on the results of /12/. Let a segment $\Gamma_0(t)$ of the separation line $\Gamma(t)$ of an elastic medium from a smooth rigid body with surface S, with smooth curvature in the small neighbourhood $\Gamma_0\ (\Gamma\in S)$, move with velocity $c(\mathbf{y}_0,t)$ $(\mathbf{y}_0\in\Gamma_0)$ relative to the elastic medium at rest, under the constraints $c_2<|c|<c_1$, and let $c(\mathbf{y}_0,t)$ have bounded derivatives in all arguments (t is time). Let us also assume that at the instant of time in question no incident short waves exist in the small elastic neighbourhood of Γ_0 (there is no diffraction of short waves on the segment Γ_0 of the singular line Γ). Then the principal term of the asymptotic form $\sigma_{nn}(\mathbf{y},t)$ and $V_n(\mathbf{y},t)-V_n(\mathbf{y}_0,t)$ (the subscript n denotes the direction along the normal to S, $\mathbf{y}\in S$, $\mathbf{y}_0\in\Gamma_0$) will be proportional to $Q(\mathbf{y}_0,t)\,|\mathbf{y}-\mathbf{y}_0|^\gamma$ as $\mathbf{y}\to\mathbf{y}_0$ where $Q(\mathbf{y}_0,t)$ is the coefficient common to $\sigma_{nn}(\mathbf{y}_0,t)$ and $V_n(\mathbf{y},t)-V_n(\mathbf{y}_0,t)$, depending on the problem as a whole /12/ $(Q(\mathbf{y}_0,t)\equiv G(l)$ for the problem discussed in Sect.1–4). In addition, using the results obtained in /12/ we can prove.

Assertion 3. With the assumptions made above, the conditions of attachment and impermeability analogous to (1.3) and (1.4) will hold if and only if

$$Q(\mathbf{y}_0,t)=0 \tag{6.1}$$

Assertion 4. Condition (6.1) is equivalent to the condition that the curvature of the free stream line is bounded on Γ_0. It appears possible to extend the above results to the case when friction of varying physical nature is present. Zviagin* has nearly succeeded in proving the validity of the Brillouin-Villat condition in the super-Rayleigh subsonic flow of an elastic medium past thin solids, and in fact used it in specific examples (to complete the proof it lacked only the condition of impermeability (1.4)). In /2/ l was determined in the same case by using the condition for a small plastic zone to form in front of the wedge. Introducing dry or viscous friction does not introduce any fundamental changes in the solution of the problem formulated in Sect.1. In the case of dry friction (we will discuss the cases separately) the nature of the singularity at zero is altered: the singularity will be a power quantity, the power index will depend on the value of c, and will be positive for some values of c (weak singularity) and negative for others. Generally speaking, the range of velocities $c_2<c<c_1$ can be divided into three subranges. In each of these subranges and on the boundaries separating them the solution will have its set of singularities at the singular points, and this makes the presentation of the problem quite complicated. the analysis of the separation is analogous to the case discussed above, and the conclusions obtained are analogous to Assertions 1 and 2 for the two extreme velocity ranges. In the middle velocity range the flow can become detached only at the trailing edge of the blade and the stresses and velocities will be unbounded at the separation point (power singularity).

We note, however, that the influence of friction can be small (the coefficient of friction can be small, the tangential stresses are limited by the condition of yield, etc.). In this case the tangential stresses at the contact surface can be taken into account to a first approximation using the solution obtained as the zeroth approximation.

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INTEGRAL ESTIMATE OF THE PRESSURE IN AN INCOMPRESSIBLE MEDIUM*

IA. A. KAMENIARZH

In the case of incompressible media the problems of equilibrium or of slow steady motion can, in many instances, be formulated without taking the pressure into account. The resulting "deviator" problem is usually easier to tackle, but it yields the stress deviator field τ only. The question arises in this connection of the possibility of returning to the initial formulation, i.e. of supplementing τ by a pressure field p, such that the condition of equilibrium with given volume and surface forces will hold for the stresses $\sigma=\tau+pg$ (g is the metric tensor). Since the general assertions do not, as a rule, guarantee the smoothness of τ, the problem needs special attention. In particular, an estimate of the pressure when the corresponding data on the stress deviator are available, is of interest. The estimate

$$\int_\Omega |p|^r dx \leqslant c \int_\Omega |\tau|^r dx \quad (1<r<\infty)$$

obtained in the paper generalizes the analogous result known for $r=2$ [1, 2]. This jusitifes the elimination of the pressure from a number of problems. Moreover, the estimate obtained can be applied directly to, for example, the pressure in perfectly plastic and viscoelastic bodies. Sect.1 gives an exact formulation of the problem and quotes examples of the cases for which it is of interest. The fundamental result is given in Sect.2 and proved in Sect.4 after establishing in Sect.3 the assertions used in the proof and concerning the fields with prescribed divergence, and re-establishment of the distribution over the derivatives. Finally, Sect.5 gives assertions facilitating the confirmation, for any problem, of the conditions under which the fundamental result was obtained.

1. Examples. Formulation of the problem. Before producing the exact formulation of the problem, we will consider several examples.

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